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Joint probability of instants of occurrence of photon events and estimation of optical parameters

Bahaa E A Saleh†

Electrical Engineering Department, University of Santa Catarina, Florianopolis, Santa Catarina, Brazil

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Abstract. Using a Karhunen–Loève modal expansion of a stationary and gaussian optical field, an expression is derived for the joint probability of instants of occurrence of photon events. When the observation time is much longer than the coherence time, this expression takes a simple form. In this case, and for lorentzian light, it is shown that measurement of the total number of counts is sufficient to determine the maximum likelihood estimate (MLE) of the average count rate. A nonlinear equation is found whose solution gives the MLE of the bandwidth.

1. Introduction

The techniques of photon counting are gaining great importance in several fields among which are optical communications (see eg Karp *et al* 1970), spectroscopy (see eg Jakeman 1974) and image restoration (see eg Helstrom 1972 and Amoss and Davidson 1972). The most popular method of counting is to divide the available observation time interval $[0, T]$ into subintervals and to find the number of counts in each.

These numbers represent the observation on which the detection of light signals or the estimation of its parameters is based. However, it has been argued by Bar-David (1969) that this counting scheme does not, in principle, retrieve all the information contained in the observed photons. The reason is that there is more than one way of subdividing the observation time, and in order to retrieve the complete information, photons should be counted in all possible subdivisions of $[0, T]$.

Another more direct, but less popular, approach is to measure the set of times at which photons occur. This set naturally contains all the information carried by the arriving photons in $[0, T]$, and estimates based on it should be superior, or equivalent, to those based on counting in a particular set of subintervals. The statistical detection and estimation based on this method have been studied by Bar-David (1969) who assumed the optical field to be coherent. More recent studies (Lutz 1970, Davidson and Amoss 1973) have been concerned with estimations based on the arrival time of the mth photo-event. When $m = 1$, this is equivalent to the method of instants of occurrence which is the most accurate. Davidson and Amoss reached this logical conclusion by computing the estimated accuracy for different values of m.

They also concluded that if this method is used to estimate the average count rate of gaussian thermal light it gives estimates that are more accurate than those calculated by the regular method of counting in subintervals. This comparison has also been done

† Now at Max-Planck-Institut für Biophysikalische Chemie, Göttingen, Germany.

by El-Sayyad (1972) on a more rigorous statistical basis. He finds that the best scheme depends on the loss function, the prior distribution, the cost of time and cost of sampling.

Estimates based on time measurements cannot be correctly found without the knowledge of the joint probability distribution of instants of occurrence of photo-events. An expression for this probability has been found for coherent light (Bar-David 1969). Davidson and Amoss (1973) have found an expression for the probability density of the time of arrival of the mth photon for thermal light. However, they avoided the problem of finding the joint probability of several arrivals by waiting after each measurement for a time period longer than the light coherence time. This ensured that the measurements are independent but, of course, part of the available observation time is not utilized. The author of this paper has found a simple expression for the probability density of the arrival time in terms of the moment generating function of the intensity fluctuations of a general gaussian light (Saleh 1973).

The object of the present paper is twofold. The first is to find a general expression for the joint probability of instants of occurrence of photon events for a general gaussian stationary and spectrally pure optical field. The second is to discuss the possibility of finding maximum likelihood estimators (MLE) based on such observations for the average light intensity and the bandwidth.

A classical, instead of quantum mechanical, analysis is followed throughout this paper. Quantum mechanically, our joint probability can be written as a trace over the normally ordered field operators. With the common coherent-state representation of the field's density operator, the trace can be shown to be equivalent to the classical ensemble average of the analogous classical field variables.

One limitation on the model we adopt is that we assume that it is possible to measure the instants of photon events accurately. This assumption is necessary to see the value of the model before attempts are made to account for such effects.

2. The joint probability of times of occurrence of photon events

An optical field represented by its analytic signal $V(\boldsymbol{r}, t)$ is detected in the time interval $[0, T]$ by a photodetector of area A. A total number of S photons is observed. They occur at times $\{t_1, t_2, \ldots, t_S\}$. The conditional joint probability density of obtaining $\{t_1, \ldots, t_S\}$ given a realization of the optical field has the form (Bar-David 1969)

$$P\{t_1, \ldots, t_S | U(t)\} = \exp\left(-\int_0^T U(t)\,\mathrm{d}t\right) \prod_{i=1}^{S} U(t_i) \tag{1}$$

where

$$U(t) = \alpha \int_A |V(\boldsymbol{r}, t)|^2 \,\mathrm{d}^2\boldsymbol{r}$$

and α is the detector quantum efficiency divided by the energy in a quantum of light. For simplicitly, we define the field

$$E(\boldsymbol{r}, t) = \surd(\alpha) V(\boldsymbol{r}, t),$$

and thus have

$$U(t) = \int_A |E(\boldsymbol{r}, t)|^2 \,\mathrm{d}^2\boldsymbol{r}. \tag{2}$$

The field $E(\boldsymbol{r}, t)$ is assumed to be a stationary and spectrally pure complex circular gaussian stochastic process. The process $U(t)$ is also stationary and has an expectation which equals the average counting rate $\bar{n}$.

The joint density of obtaining $\{t_1, \ldots, t_S\}$ is obtained by averaging (1) over the realizations of $U(t)$, thus

$$P\{t_1, \ldots, t_S\} = \left\langle \exp\left(-\int_0^T U(t)\,\mathrm{d}t \right) \prod_{i=1}^{S} U(t_i) \right\rangle . \tag{3}$$

We are interested in finding an expression for this expectation. For this purpose we use a modal expansion (Helstrom 1970, Kelly 1972) for the process $E(\boldsymbol{r}, t)$ which is a temporal–spatial generalization of the Karhunen–Loève expansion (see eg Van Trees 1968). Thus,

$$E(\boldsymbol{r}, t) = \sum_p \sum_m a_{pm} f_{pm}(\boldsymbol{r}, t), \tag{4}$$

where the coefficients a_{pm} are statistically independent complex random variables. The expansion functions $f_{pm}(\boldsymbol{r}, t)$ are orthonormal over the interval $[0, T]$ and the detector area A:

$$\int_A \int_0^T f^*_{pm}(\boldsymbol{r}, t) f_{qn}(\boldsymbol{r}, t)\,\mathrm{d}t\,\mathrm{d}^2\boldsymbol{r} = \delta_{pq}\delta_{mn}. \tag{5}$$

This is possible if $f_{pm}(\boldsymbol{r}, t)$ are the eigenfunctions of the correlation function

$$G(\boldsymbol{r}, t; \bar{\boldsymbol{r}}, \bar{t}) = \langle E^*(\boldsymbol{r}, t) E(\bar{\boldsymbol{r}}, \bar{t}) \rangle, \tag{6}$$

that is,

$$\int_A \int_0^T G(\boldsymbol{r}, t; \bar{\boldsymbol{r}}, \bar{t}) f_{pm}(\boldsymbol{r}, t)\,\mathrm{d}t\,\mathrm{d}^2\boldsymbol{r} = \lambda_{pm} f_{pm}(\bar{\boldsymbol{r}}, \bar{t}). \tag{7}$$

If the field is spectrally pure, (6) can be written as

$$G(\boldsymbol{r}, t; \bar{\boldsymbol{r}}, \bar{t}) = G(\boldsymbol{r}, \bar{\boldsymbol{r}}) \chi(t - \bar{t}) \exp \mathrm{i}\Omega(t - \bar{t}) \tag{8}$$

where Ω is the central frequency, and this permits us to break each eigenfunction $f_{pm}(\boldsymbol{r}, t)$ into spatial and temporal parts,

$$f_{pm}(\boldsymbol{r}, t) = \xi_p(\boldsymbol{r}) \gamma_m(t) \exp(-\mathrm{i}\Omega t) \tag{9}$$

$$\lambda_{pm} = h_p g_m \tag{10}$$

where $\gamma_m(t)$ are the eigenfunctions of $\chi(t - \bar{t})$,

$$\int_0^T \chi(t - \bar{t}) \gamma_m(t)\,\mathrm{d}t = g_m \gamma_m(\bar{t}), \tag{11}$$

which are orthonormal over $[0, T]$. Also, $\xi_p(\boldsymbol{r})$ are the eigenfunctions of $G(\boldsymbol{r}, \bar{\boldsymbol{r}})$

$$\int_A G(\boldsymbol{r}, \bar{\boldsymbol{r}}) \xi_p(\boldsymbol{r})\,\mathrm{d}^2\boldsymbol{r} = h_p \xi_p(\bar{\boldsymbol{r}}), \tag{12}$$

which are orthonormal over A. The eigenvalues h_p are scaled so that $\Sigma h_p = 1$.

Because the field is gaussian, the coefficients $\{a_{pm}\}$ are uncorrelated complex gaussian variables whose variances are the eigenvalues $\{\lambda_{pm}\}$ of (7) (see eg Jakeman and Pike 1968), ie

$$P(a_{pm}) = \frac{1}{\pi\lambda_{pm}} \exp\left(-\frac{|a_{pm}|^2}{\lambda_{pm}}\right). \tag{13}$$

By substituting (4) and (9) in (2) and using the orthonormality of $\xi_p(\boldsymbol{r})$, we can write

$$U(t) = \sum_p \sum_{m,n} a^*_{pm} a_{pn} \psi_{mn}(t) \tag{14}$$

where

$$\psi_{mn}(t) = \gamma^*_m(t)\gamma_n(t). \tag{15}$$

Also, by using (14) and the orthonormality of $\{\gamma_m(t)\}$ we get

$$\int_0^T U(t)\,\mathrm{d}t = \sum_p \sum_m |a_{pm}|^2. \tag{16}$$

Now we are in a position to find out the expectation in (3).

For simplicity, we start with the special case $S = 1$. By substituting from (14) and (16) in (3) we obtain

$$P(t_1) = \sum_p \sum_{m,n} C_{p,mn} \psi_{mn}(t) \tag{17}$$

where

$$C_{p,mn} = \left\langle a^*_{pm} a_{pn} \exp\left(-\sum_{\bar{p}} \sum_{\bar{m}} |a_{\bar{p}\bar{m}}|^2\right)\right\rangle.$$

Noting that $\{a_{pm}\}$ are independent and using (13), it follows that

$$C_{p,mn} = Q(1)\bar{\lambda}_{pm}\delta_{mn} \tag{18}$$

where

$$\bar{\lambda}_{pm} = \frac{\lambda_{pm}}{1+\lambda_{pm}} \tag{19}$$

and

$$Q(1) = \prod_{p,m} \frac{1}{1+\lambda_{pm}}. \tag{20}$$

The function

$$Q(s) = \prod_{p,m} (1+s\lambda_{pm})^{-1}$$

is known (Jakeman and Pike 1968) to be the moment generating function of the process $\int_0^T U(t)\,\mathrm{d}t$. Expressions for this function have already been found for most fields of practical importance (Jakeman and Pike 1969). With the substitution of (18) in (17), we get the desired probability density as a function of the eigenvalues and eigenfunctions of the mutual coherence function,

$$P(t_1) = Q(1) \sum_{p,m} \bar{\lambda}_{pm} \psi_{mm}(t_1).$$

From (10) it follows that

$$P(t_1) = Q(1)\sum_m \bar{g}_m \psi_{mm}(t) = Q(1)\sum_m \bar{g}_m |\gamma_m(t_1)|^2 \tag{21}$$

where

$$\bar{g}_m = \sum_p \frac{\lambda_{pm}}{1+\lambda_{pm}} = g_m \sum_p \frac{h_p}{1+h_p g_m}. \tag{22}$$

Since the field is stationary, this probability should be uniform over the interval $[0, T]$.

The derivation of an expression for the joint probability in the general case is a straightforward generalization of the above derivation. This leads to the general equation

$$P(t_1, \ldots, t_S) = Q(1)\sum_{m_1}\sum_{m_2}\cdots\sum_{m_S} \bar{g}_{m_1}\bar{g}_{m_2}\cdots\bar{g}_{m_S}\left(\sum_{\pi} \psi_{m_1\pi_1}(t_1)\psi_{m_2\pi_2}(t_2)\ldots\psi_{m_S\pi_S}(t_S)\right) \tag{23}$$

where Σ_π is the summation over the permutations of $(m_1, m_2, \ldots, m_S)$. As an example, we write (23) for $S = 2$,

$$\begin{aligned} P(t_1, t_2) &= Q(1)\sum_m\sum_n \bar{g}_m\bar{g}_n[\psi_{mm}(t_1)\psi_{nn}(t_2)+\psi_{mn}(t_1)\psi_{nm}(t_2)] \\ &= Q(1)\left(\sum_m \bar{g}_m|\gamma_m(t_1)|^2\sum_n \bar{g}_n|\gamma_n(t_2)|^2 + \left|\sum_m \bar{g}_m\gamma_m^*(t_1)\gamma_m(t_2)\right|^2\right). \end{aligned} \tag{24}$$

This probability should be a function of $(t_2 - t_1)$. Equation (23) is one of the main results of this paper. If the sets $\{g_m\}$, $\{h_p\}$, $\{\gamma_m(t)\}$ and $Q(1)$ are known for a particular field, the joint probability of $\{t_1, t_2, \ldots, t_S\}$ can be directly obtained. In many practical situations, the area of the detector is much smaller than the coherence area of the field. In this case the optical field can be approximately treated as spatially coherent, which means it has only one spatial mode. In this case, we have one spatial eigenvalue $h_p = 1$ and hence

$$\bar{g}_m = \frac{g_m}{1+g_m}. \tag{25}$$

On the other hand, the observation time is usually longer than the coherence time and therefore we generally have a larger number of temporal modes. The eigenvalue problem (11), has been solved for several important distributions of $\chi(t-\bar{t})$. In such cases, results can be substituted in (23) and a lengthy expression for the desired joint probability will be obtained. But if the observation time covers a very large number of coherence times, the results of this section can be greatly simplified.

3. Approximations based on long observation time

If the observation time T is much longer than the field's coherence time τ_c, then the eigenvalues and eigenfunctions in (11) approximately take the simple forms (Van Trees 1968)

$$\begin{aligned} &\gamma_m(t) \simeq (1/\sqrt{T})\exp(\mathrm{i}\omega_m t), \qquad && \omega_m = m(2\pi/T) \\ &g_m \simeq X(\omega_m), && m = 0, \pm 1, \pm 2, \ldots \end{aligned} \tag{26}$$

where

$$X(\omega) = \int_{-\infty}^{\infty} \chi(t) \exp(-\mathrm{i}\omega t)\,\mathrm{d}t \tag{27}$$

is the power density spectrum. Also

$$\sum_m Y(g_m) \simeq \frac{T}{2\pi}\int_{-\infty}^{\infty} Y[X(\omega)]\,\mathrm{d}\omega \tag{28}$$

where Y is any function of g_m. In practical situations T/τ_c is large enough to justify the above approximation (Helstrom 1970). By applying (28) to (21) and using (26), we get

$$P(t_1) \simeq Q(1)\int_{-\infty}^{\infty} \frac{X(\omega)}{X(\omega)+1}\frac{\mathrm{d}\omega}{2\pi} \triangleq Q(1)\eta(0) \tag{29}$$

which is independent of time as it should be. Applying (28) again to (24), we get

$$P(t_1, t_2) = Q(1)[\eta^2(0) + |\eta(t_2 - t_1)|^2] \tag{30}$$

where

$$\eta(\tau) = \int_{-\infty}^{\infty} \frac{X(\omega)}{X(\omega)+1} \mathrm{e}^{\mathrm{i}\omega\tau}\frac{\mathrm{d}\omega}{2\pi}. \tag{31}$$

In general, we apply (28) to (23) and with some algebraic manipulations obtain

$$P(t_1, \ldots, t_S) = Q(1)\sum_\pi \eta(t_1 - t_{\pi 1})\eta(t_2 - t_{\pi 2}) \ldots \eta(t_S - t_{\pi S}), \tag{32}$$

where π denotes permutations over $(1, 2, \ldots, S)$.

Equation (32), together with (31), is indeed a simple expression for the joint probability as a function of the field's power density spectrum. It is obviously stationary and it satisfies the symmetries of a joint probability function.

3.1. *Lorentzian light*

A very important class of optical field is that which has a lorentzian spectrum. For such a field, $\chi(\tau) = \bar{n}\exp(-\Gamma|\tau|)$ and

$$X(\omega) = \bar{n}\frac{2\Gamma}{\omega^2 + \Gamma^2},$$

from which

$$\eta(\tau) = \frac{\bar{n}}{(1 + 2(\bar{n}/\Gamma))^{1/2}} \exp\left[-\Gamma\left(1 + 2\frac{\bar{n}}{\Gamma}\right)^{1/2}|\tau|\right]. \tag{33}$$

Also, $Q(1)$ has been shown to have the form (Jakeman and Pike 1968)

$$Q(1) = 2\exp\left(\Gamma T\left[1 - \left(1 + 2\frac{\bar{n}}{\Gamma}\right)^{1/2}\right]\left\{1 + \frac{1}{2}\left[\left(1 + 2\frac{\bar{n}}{\Gamma}\right)^{1/2} + \frac{1}{(1 + 2(\bar{n}/\Gamma))^{1/2}}\right]\right\}^{-1}\right) \tag{34}$$

where ΓT has been assumed very large. With the substitution of (38) and (34) in (32), we get an expression for the joint probability as a function of the instants of occurrence and the parameters $\bar{n}$ and Γ describing the field.

Let us apply this to the special case of very weak light ($\bar{n}/\Gamma \ll 1$). In this case, $\eta(\tau) \simeq \bar{n}\exp(-\Gamma\tau) \simeq \chi(\tau)$ and $Q(1) \simeq \exp(-\bar{n}T)$. By substituting this in (32) and using the expansion of intensity coherence functions of gaussian light (Reed 1962), we get

$$P(t_1, \ldots, t_S) \simeq \bar{n}^S \, \mathrm{e}^{-\bar{n}T} g(t_1, \ldots, t_S) \tag{35}$$

where

$$g(t_1, \ldots, t_S) = \frac{\langle \Pi_{i=1}^{S} |E(t_i)|^2 \rangle}{\bar{n}^S}$$

is the normalized Sth order intensity coherence function. Equation (35) can be obtained directly from (3), simply by using the approximation $T^{-1}\int_0^T U(t)\,\mathrm{d}t \simeq \bar{n}$. Other results based on this approximation have also been obtained by Saleh (1974).

4. Estimation of average count rate and bandwidth of gaussian–lorentzian light

In this section, we deal with gaussian–lorentzian light observed in a time T much longer than the coherence time τ_c. Hence the joint probability of photon instants is given by (32) together with (33) and (34). It depends on two parameters, the average count rate $\bar{n}$ and the bandwidth Γ. Given the observed times $(t_1, \ldots, t_S)$ we are interested in finding the MLE for $\bar{n}$ and Γ. These are the values of $\bar{n}$ and Γ which make the joint probability of the observations a maximum.

For simplicity, we introduce two new positive variables

$$\beta = +\Gamma T\left(1+2\frac{\bar{n}}{\Gamma}\right)^{1/2} \gg 1, \tag{36}$$

and

$$\epsilon = +\left(1+2\frac{\bar{n}}{\Gamma}\right)^{1/2} > 1. \tag{37}$$

Using (25), (26) and (27), we can write the joint probability as a function of β and ϵ, thus

$$P(t_1, \ldots, t_S) = \frac{2^{S-1}}{T^S}\left(1-\frac{1}{\epsilon^2}\right)^S\left[1+\frac{1}{2}\left(\epsilon+\frac{1}{\epsilon}\right)\right]^{-1} \mathrm{e}^{\beta/\epsilon}\,\mathrm{e}^{-\beta}\beta^S \sum_{\pi} \mathrm{e}^{-\beta\{\ \}_\pi} \tag{38}$$

where

$$\{\ \}_\pi = \{|t_1-t_{\pi 1}|+|t_2-t_{\pi 2}|+\ \ldots\ +|t_S-t_{\pi S}|\}. \tag{39}$$

There exists a one to one correspondence between the variables $(\bar{n}, \Gamma)$ and (ϵ, β) determined by the transformation (36) and (37). Hence we can use the invariance property of MLE (Kendall and Stuart 1973) and determine ϵ and β which maximize (38), from which we obtain $\bar{n}$ and Γ. Let us define a likelihood function proportional to the logarithm of the joint probability

$$L(t_1, \ldots, t_S; \epsilon, \beta) = S\ln\left(1-\frac{1}{\epsilon^2}\right) - \ln\left[1+\frac{1}{2}\left(\epsilon+\frac{1}{\epsilon}\right)\right] + \frac{\beta}{\epsilon} + S\ln(\beta) + \ln\left(\sum_{\pi}\mathrm{e}^{-\beta\{\ \}_\pi}\right).$$

To find the maximum of this function, we equate the first derivatives with respect to ϵ and β to zero and get

$$\beta = \epsilon\frac{2S-(\epsilon-1)^2}{\epsilon^2-1}, \tag{40}$$

and

$$\frac{1}{\epsilon}+\frac{S}{\beta}-1 = \frac{\Sigma_\pi\{\ \}_\pi \mathrm{e}^{-\beta\{\ \}_\pi}}{\Sigma_\pi \mathrm{e}^{-\beta\{\ \}_\pi}}. \tag{41}$$

These two equations should be solved together for ϵ and β. But let us first examine (40). By some algebraic manipulations, it can be shown that it is equivalent to

$$\bar{n} = \frac{S}{T}-\frac{1}{T}\left(1+\frac{1}{\Gamma T}\right)^{-1}+\left\{\left[\frac{S}{T}-\frac{1}{T}\left(1+\frac{1}{\Gamma T}\right)^{-1}\right]^2-\frac{S(S-2)}{T^2}\right\}^{1/2}. \tag{42}$$

Equation (42) expresses a conclusion of some importance. The MLE for $\bar{n}$ and Γ are related by an equation in which the observation S appears but the times $t_1, \ldots, t_S$ do not. Moreover, if we use again our earlier assumption $\Gamma T \gg 1$, on which (38) itself is based, we simply get

$$\bar{n} \simeq \frac{S}{T}. \tag{43}$$

Thus, the statistic S is sufficient for estimating $\bar{n}$. As far as the estimation of $\bar{n}$ is concerned, the measurement of when exactly does each of the S photons occur does not contribute any useful information. The reader should be reminded that this holds good under the assumptions of gaussian–lorentzian field with $\Gamma T \gg 1$.

We turn now to the estimation of Γ. Equation (43) is equivalent to

$$\epsilon = \frac{1}{\beta}[S+(S^2+\beta^2)^{1/2}]$$

which when substituted in (41) gives

$$\beta^2[S+(S^2+\beta^2)^{1/2}]^{-1}+S-\beta = \frac{\Sigma_\pi \beta\{\ \}_\pi \mathrm{e}^{-\beta\{\ \}_\pi}}{\Sigma_\pi \mathrm{e}^{-\beta\{\ \}_\pi}} \tag{44}$$

which is a nonlinear equation in β. Equation (44) must have a solution. The reason is evident from examination of (38). The joint probability is a continuous positive function of β which vanishes at $\beta = 0$ and $\beta = \infty$. Hence, at least one non-trivial value of β at which the derivative is zero must exist.

5. Conclusions

It has been made clear in the introduction that the MLE based on the instants of occurrence of photoevents should have better accuracy than those obtained by the autocorrelation technique (based on equating the correlation function of the sample to that of the process, and known in statistics as the method of moments). The question of exactly how much gain in accuracy is obtained by this new technique is difficult to answer, because in order to test equation (44) and the accuracy of bandwidth estimates based on it, numerical values for $(t_1, \ldots, t_S)$ must be given. Such values can be obtained either experimentally or by computer simulation of the process. A theoretical expression for the estimation accuracy is extremely difficult to obtain.

Questions may be raised about the practicability of this technique, because of the extremely large number of photoevents involved in a typical spectroscopy experiment. Computations can always be made with a computer, although it may necessitate a very lengthy computation time. This may be acceptable in certain situations. Also, there may be situations in which only a few photons arrive during the course of an experiment so that full use must be made of the arrival times, as suggested in this paper, if any information about the source is to be gained.

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